

copy 2 ①  
JUL 3 1963

# PROJECT SQUID

TECHNICAL REPORT ARAP-6-R

## THE SUPERSEULARITIES IN WEAK COUPLING KINETIC THEORY

By

Guido Sandri

Aeronautical Research Associates of Princeton, Inc.

PRINCETON UNIVERSITY  
THE JAMES FORRESTAL  
RESEARCH CENTER  
LIBRARY

DEPARTMENT OF AEROSPACE ENGINEERING  
SCHOOL OF ENGINEERING AND APPLIED SCIENCE  
UNIVERSITY OF VIRGINIA  
CHARLOTTESVILLE, VIRGINIA

Project SQUID is a cooperative program of basic research relating to Jet Propulsion. It is sponsored by the Office of Naval Research and is administered by the University of Virginia through Contract Nonr 3623(00) NR-098-038.

June 1963

DTIC  
ELECTE  
NOV 07 1983

This document has been approved  
for public release and sale; its  
distribution is unlimited.

83 11 04 069

DTIC FILE COPY

ARAP-6-R

A952 480

Technical Report ARAP-6-R

PROJECT SQUID

A COOPERATIVE PROGRAM OF FUNDAMENTAL RESEARCH  
AS RELATED TO JET PROPULSION  
OFFICE OF NAVAL RESEARCH, DEPARTMENT OF THE NAVY

Contract Nonr 3623(00), NR-098-038

THE SUPERSECULARITIES IN WEAK  
COUPLING KINETIC THEORY

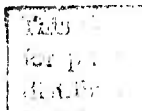
by

Guido Sandri

Aeronautical Research Associates of Princeton, Inc.  
50 Washington Road, Princeton, New Jersey

June 1963

Reproduction, translation, publication, use and disposal in  
whole or in part by or for the United States Government is permitted.



# TABLE OF CONTENTS

<u>Section</u>		<u>Page</u>
	Abstract . . . . .	v
I.	Introduction. . . . .	1
II.	Graphs . . . . .	3
III.	The Weak-Coupling Expansion . . . . .	6
	(i) Zeroth order theory . . . . .	6
	(ii) First order theory . . . . .	6
	(iii) Second order theory . . . . .	7
	(iv) Third order theory . . . . .	10
	(v) Higher order terms . . . . .	11
IV.	Conclusion . . . . .	18
	References . . . . .	19
	Appendix. . . . .	20

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	



UNANNOUNCED

## ABSTRACT

In a recent monograph we have shown that the higher order kinetic equations are divergent in an irreparable fashion. In the present paper we determine completely the singularities in the  $n$ th order  $s$ -body distribution function for the Weak-Coupling Kinetic Theory (supersecularities).

## I. INTRODUCTION

In a recent monograph (1) we have shown that the higher order kinetic equations are divergent in an irreparable fashion. In view of this result it is impossible to calculate in finite terms the departures from the already available kinetic equations (Boltzmann's for a short range gas, Landau's for a weakly coupled gas and Bogolubov's for a Debye gas).

It is the writer's thesis that the breakdown of the expansion corresponds to the impossibility of characterizing an arbitrary state of a system of  $10^{23}$  particles by means of the state of one typical (average) particle. Whence, to understand the behavior of a system with more accuracy than that afforded by the already available kinetic equations, it is necessary to determine the evolution of a typical pair of particles. Both the form of pair-kinetic equations and a method for removing the divergences (Method of Closure) have been described (2).

The purpose of the present paper is to determine completely the nature of the singularities arising in the kinetic expansion. Thus, we shall present the explicit form of the  $n$ th order  $s$ -body distribution function by means of a graphical technique.

The singularities discussed here lead to a "super-secular" behavior in all the  $s$ -body distributions of the weak-coupling kinetic theory.

The notation to be employed is as follows. The Bogolubov expansion of the time derivative

$$D = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots \quad (1)$$

is represented by means of the independent "time" variables  $\tau_0, \tau_1, \dots, \tau_n$  by:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \dots \quad (2)$$

The hierarchy of equations for the time evolution of the s-body clusters in a weakly-coupled gas is written as:

$$\frac{\partial F^s}{\partial t} + K^s F^s = \varepsilon I^s F^s + \varepsilon L^s F^{s+1} \quad (3)$$

A table of the symbols employed is given in the Appendix.

We shall consider only spatially homogeneous gases, whence

$$K^1 F^1 = 0 \quad (4)$$

The extended (3) distribution functions are expanded by means of:

$$F^1 = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots \quad (5)$$

$$F^s = F^{s0} + \varepsilon F^{s1} + \varepsilon^2 F^{s2} + \dots \quad s \neq 1 \quad (6)$$

The special notation emphasizes the preferred role of the one-body distribution function in kinetic theory.

We shall consider only solutions of (3) that correspond to the simple initial value problem defined by

$$\frac{\partial F^s[0]}{\partial \varepsilon} = 0 \quad (7)$$

and by the requirement that all initial correlations vanish. Thus, for example,

$$f_1[0] = 0, \quad F^{21}[0] = 0$$

We shall make extensive use of the operator valued distribution

$$\bar{z}^*(iK^s) = \int_0^\infty e^{-K^s \lambda} d\lambda \equiv \bar{z}^{*s} \quad (8)$$

For compactness, we omit all particle indices whenever confusion cannot arise. Superscripts refer to the number of particles, subscripts to the particle variables.

## II. GRAPHS

The ingredients of the kinetic expansion are  $f^0$ ,  $I$ ,  $L$ , and  $\sum^*$ . These are represented as follows:

Vertical oriented lines represent  $f^0$  and  $\sum^* f_0$ , thus:

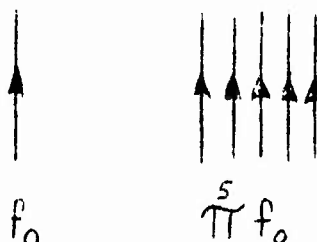


Figure 1

The "propagation functions"  $\sum^{*s}$  are denoted by curved oriented lines. Thus,

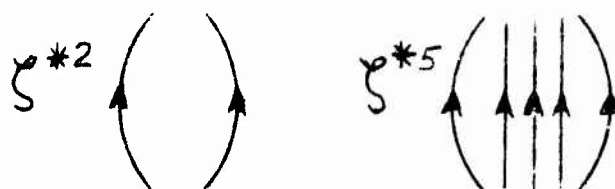


Figure 2

The "interactions"  $I^s$  are represented by circles. Thus,  $\sum^{*2} I^2 f_0 f_0$  is represented by

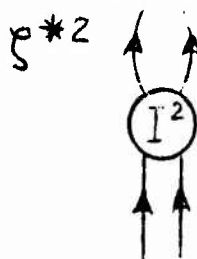


Figure 3

The "phase mixing" operator  $L = \int d\vec{x}' d\vec{v}' \vec{\nabla} U(\vec{x} - \vec{x}') \cdot \vec{\nabla}_{\vec{v}}$

is represented by a rectangle, a cross indicating the phase mixed variable. Thus,  $L_{12} \sum_{12}^* I_{12} f_{01} f_{02}$  is represented by

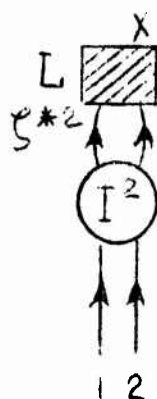


Figure 4

The skeleton of a graph  $\Sigma(G)$  is the graph with all freely streaming particles omitted. Thus, for example:

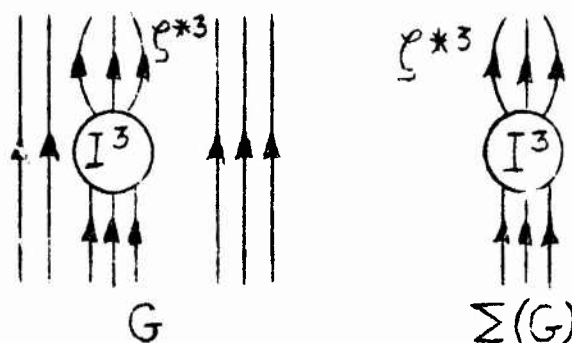


Figure 5

The essential point of our analysis is that, for a given order of approximation,  $n$ , the number of skeleton graphs,  $\mathcal{G}$ , is independent of  $s$ :

$$\frac{\partial \mathcal{G}(n)}{\partial s} = 0 \quad (9)$$

Thus, a fixed number of skeletons characterizes the order of approximation for all the clusters of particles.



We shall see that

$$\sigma(0) = \sigma(1) = 1$$

(10)

$$\sigma(2) = 4$$

$$\sigma(3) = 32$$

### III. THE WEAK-COUPPLING EXPANSION

#### (1) Zeroth order theory

From (3) we have

$$f_0 = \text{const} \quad (11)$$

$$F^{s0} = \prod^s f_0 \quad (12)$$

#### (11) First order theory

Since

$$\frac{\partial f_0}{\partial \tau_i} = 0, \quad f_i = 0 \quad (13)$$

and

$$L_s \prod^{sH} f_0 = 0 \quad (14)$$

which corresponds to the skeleton in Fig. 6,

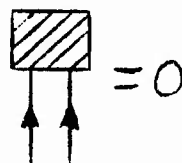


Figure 6

we have, from (3)

$$\frac{dF^{s1}}{d\tau_0} = I^s F^{s0} \quad (15)$$

with

$$\frac{d}{d\tau_0} F^{sK} = \left( \frac{\partial}{\partial \tau_0} + K^s \right) F^{sK} \quad (16)$$

Therefore,

$$F_{T_0}^{S1} \sim \sum_{i,j}^S \bar{I}^S I^S \prod f_0 = \sum_{i,j}^S (\bar{I}^* I)_{ij} \prod f_0 \quad (17)$$

We call the first expression reducible, the second irreducible (i.e., only two-body interactions appear explicitly). The reducible graph  $G$  is a sum of irreducible ones  $G^*$  corresponding to a single skeleton thus

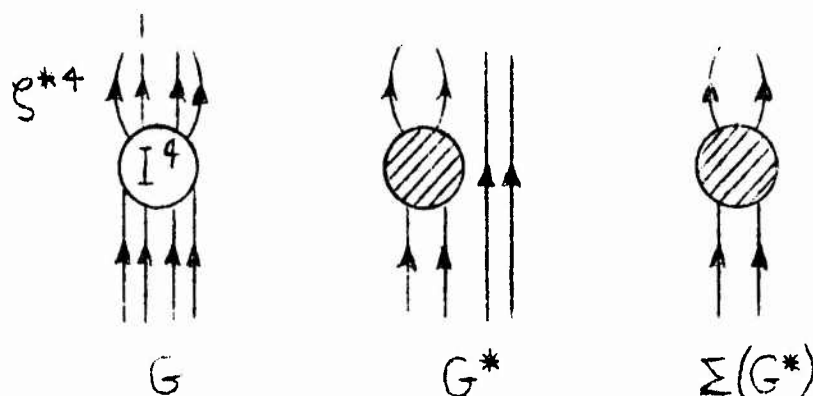


Figure 7

The  $\Sigma(G^*)$  of Fig. 7 is the only irreducible skeleton in first order theory.

### (iii) Second order theory

From (3) we have

$$\frac{dF^{S2}}{d\tau_0} = -\frac{\partial F^{S0}}{\partial \tau_2} + L_S F^{S+1,1} + I^S F^{S1} \quad (18)$$

For  $s = 1$ , (18) is the Landau equation

$$\frac{\partial f_0}{\partial \tau_2} = L (\bar{I}^* I)^2 f_0 f_0 \quad (19)$$

represented in Fig. 8. But,

$$\frac{\partial F^{S0}}{\partial \tau_2} = \sum_K^S L_{K^{S+1}} (\bar{I}^* I)_{K^{S+1}j} \prod f_{ij} \quad (20)$$

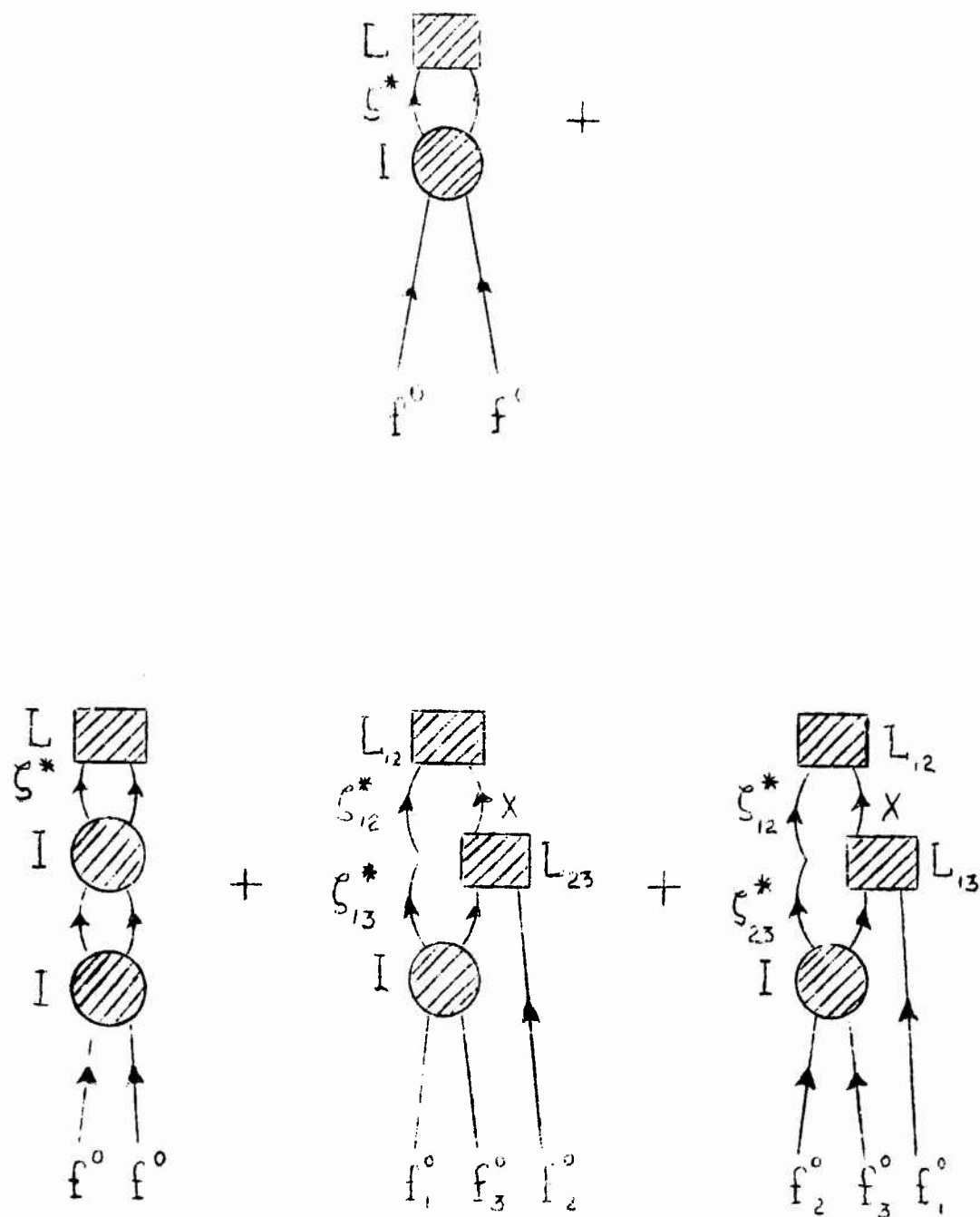


Figure 8

$$L_S F^{s+1,l} = \sum_{p=1}^S L_{p \leq s+1} \sum_{i < j}^{s+1} (\mathcal{Z}^* I)_{ij} \prod_K^{s+1} f_{0K} \quad (21)$$

(21) contains the three skeletons of Fig. 9:

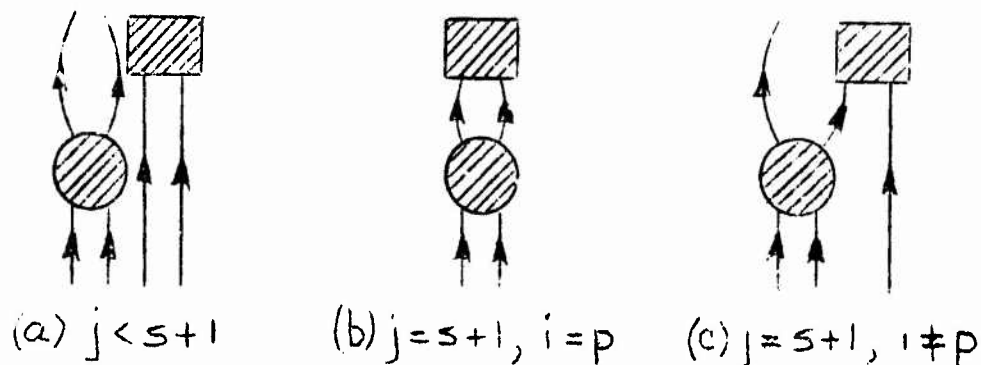


Figure 9

Skeleton (a) vanishes. (b) cancels exactly the contribution from (20). This latter cancellation of  $\partial F^{s0} / \partial \tau_n$  against a subset of  $L_S F^{s+1, n-1}$  occurs to all orders. The contribution of the last term of (18),

$$I^S F^{s1} = I^S (\mathcal{Z}^* I)^S F^{s0} \quad (22)$$

is represented by three skeletons:

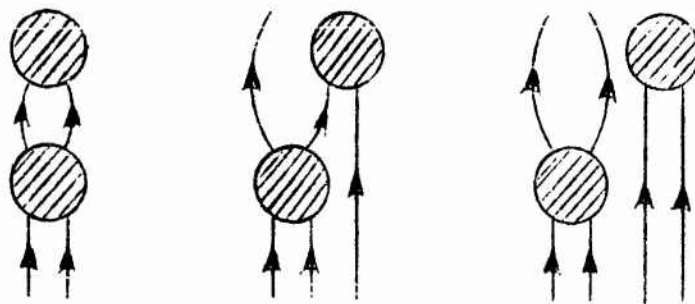


Figure 10

Adding (20), (21), and (22), and integrating (18) we find:

$$F^{S2} \sim (\tilde{S}^* I)^S (\tilde{S}^* I)^S F^{S0} + \tilde{S}^* \sum_K^{S+1} L_{K+1} \sum_{j \neq K}^S (\tilde{S}^* I)_{j+1} \prod_i^{S+1} f_{0i} \quad (23)$$

corresponding to the four skeletons

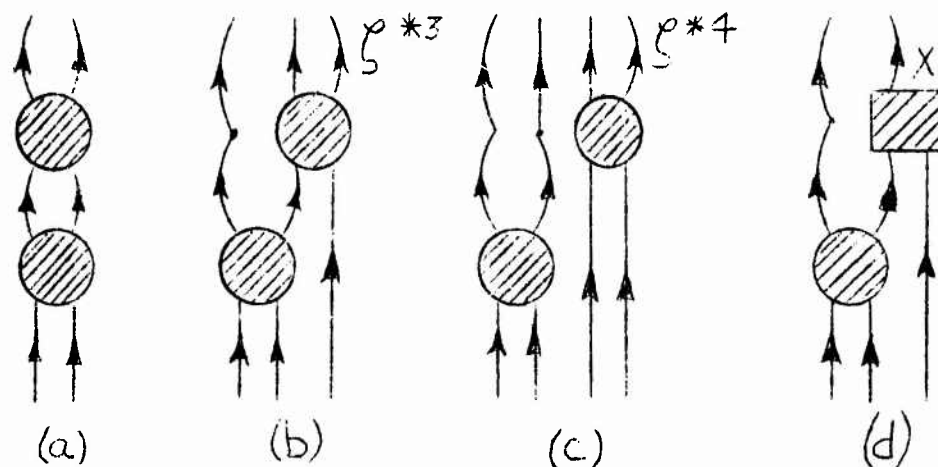


Figure 11

Clearly (b) contributes for  $S \geq 3$  only and (c) for  $S \geq 4$  only.

Note that the graphs for the asymptotic value of  $F^{S2}$  differ from those corresponding to the graphs for its  $\tau_0$  derivative only because of the propagators at the top.

#### (iv) Third order theory

At this order the expansion breaks down. From (3) we have:

$$\frac{dF^{S3}}{d\tau_0} = -\frac{\partial F^{S1}}{\partial \tau_2} - \frac{\partial F^{S0}}{\partial \tau_3} + L_S F^{S+1,2} + I^S F^{S2} \quad (24)$$

The analysis proceeds as in second order. The graphs for the kinetic condition  $\partial f^0 / \partial \tau_3$  are given in Fig. 8. The contributions from  $\partial F^{S0} / \partial \tau_3$  exactly cancel a subset of  $L_S F^{S+1,2}$ . The divergent graphs arise from  $\partial F^{S1} / \partial \tau_2$ .

The operation  $\partial/\partial\alpha_2$  in fact inserts the second order kinetic condition on each of the "free legs"  $f_0$  of the graphs for  $F^{s1}$ , leading to the following two skeletons in  $F^{s3}$ :

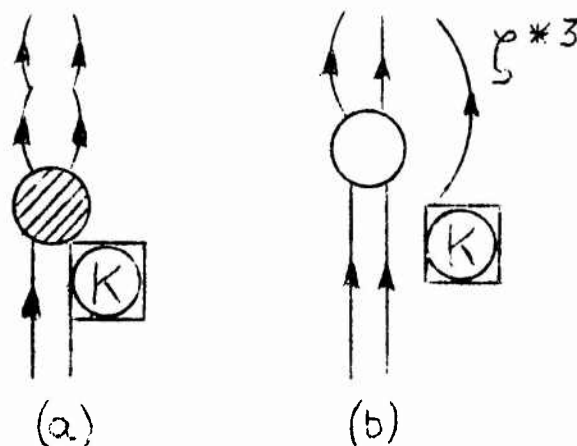


Figure 12

where (K) denotes the entire Landau graph. (a) is divergent due to the coincidence of two singular functions  $\sum_{12}^* \sum_{12}^*$ , while (b) is finite. The 32 skeletons for  $F^{s3}$  are given in Fig. 13.

(v) Higher order terms

The construction of all the skeletons of a given order has now been made completely systematic. In Fig. 14 we give the fourth order kinetic condition. The reader can convince himself of the fact that there are singularities corresponding to arbitrary powers of  $\mathcal{S}^{*s}$  introduced by the derivatives

$$\frac{\partial F^{sK}}{\partial \alpha_n} \quad K \neq 0, n > 1 \quad (24)$$

Thus, for example, the simplest  $(\mathcal{S}^{*3})^2$  divergences are represented by the skeletons in Fig. 15.

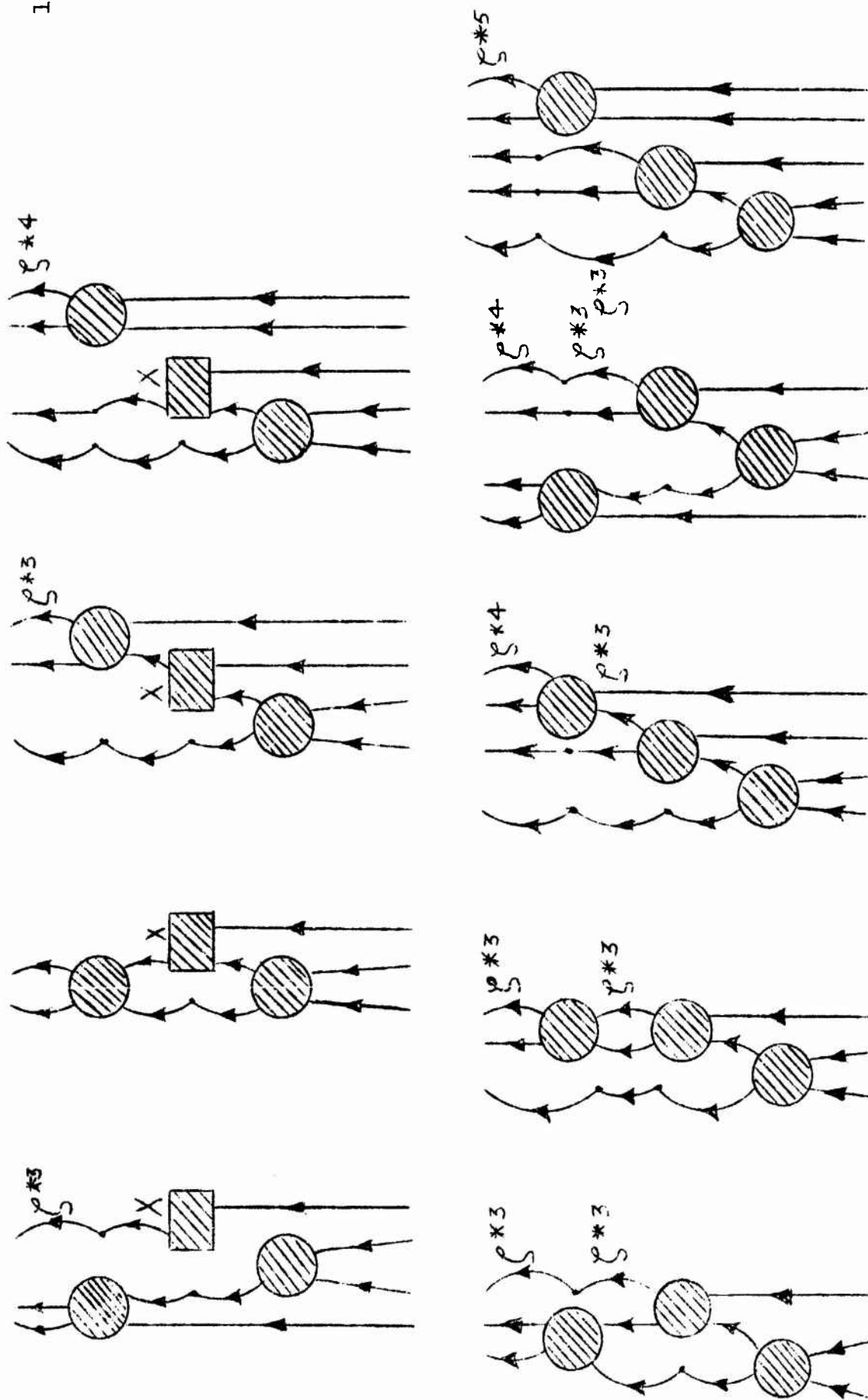


Figure 13(I)



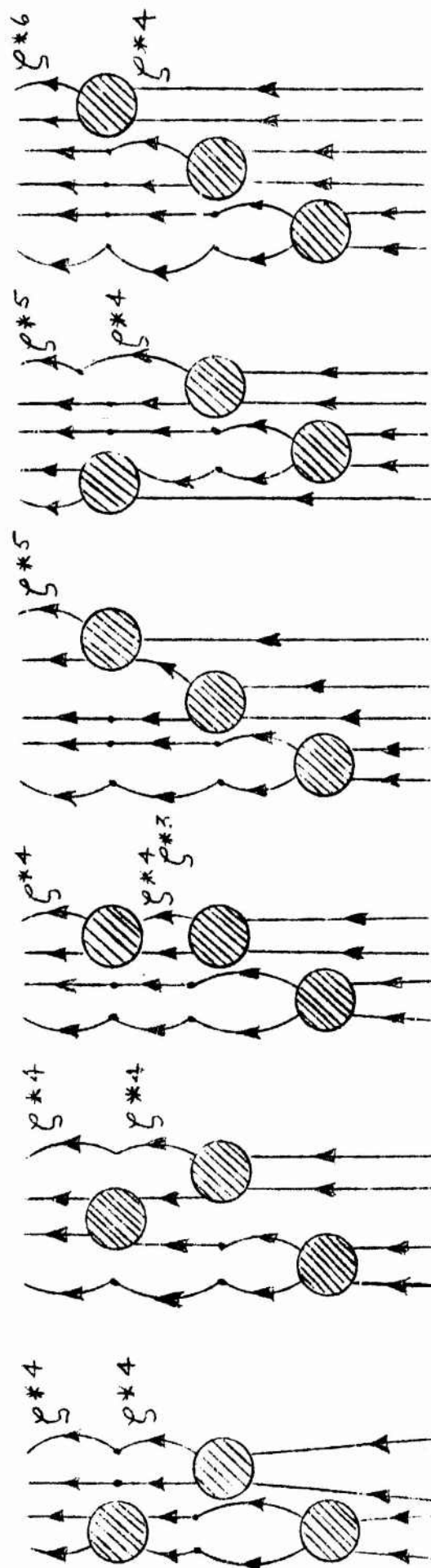
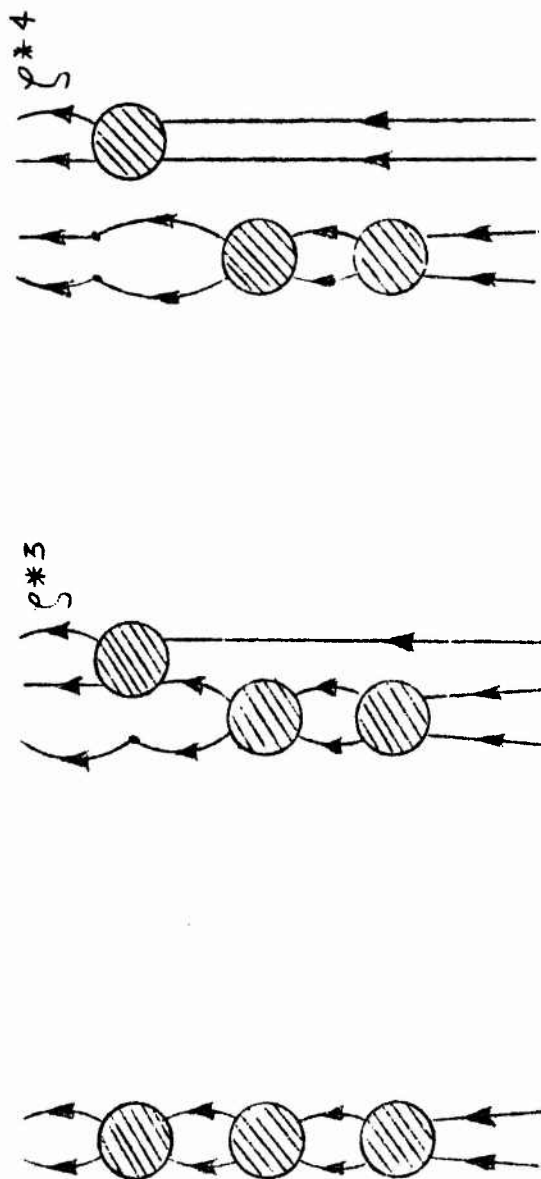


Figure 13(II)

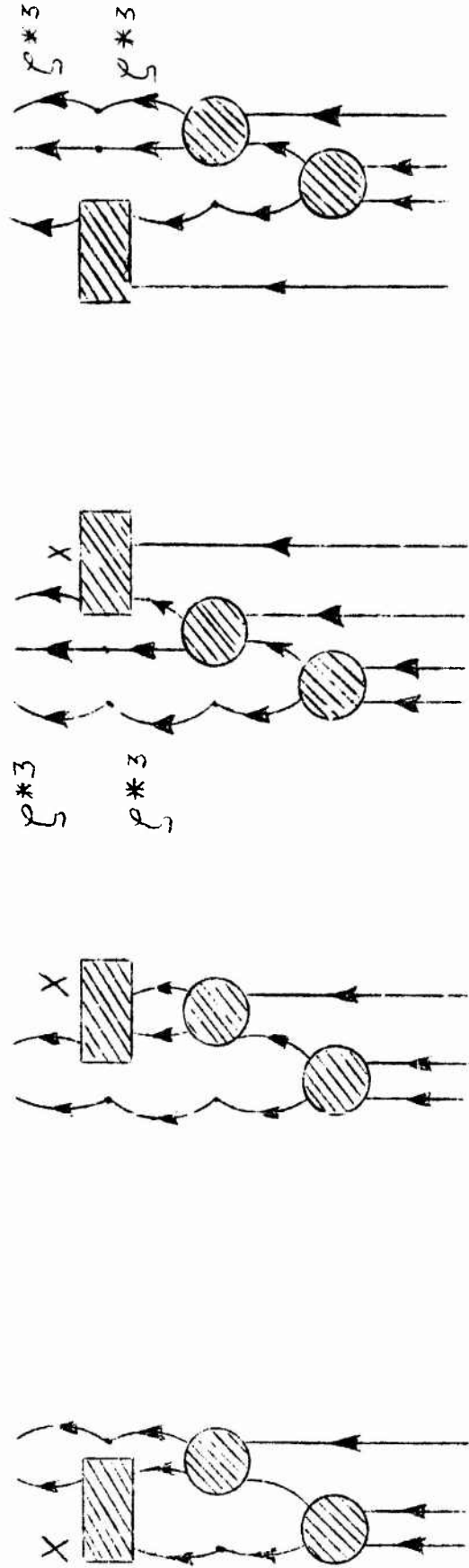
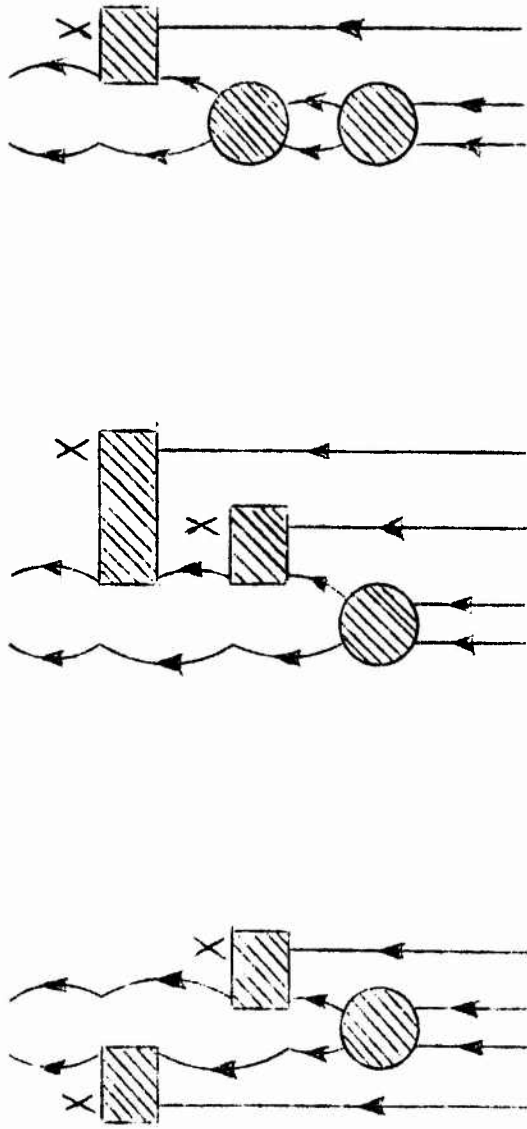


Figure 13(III)

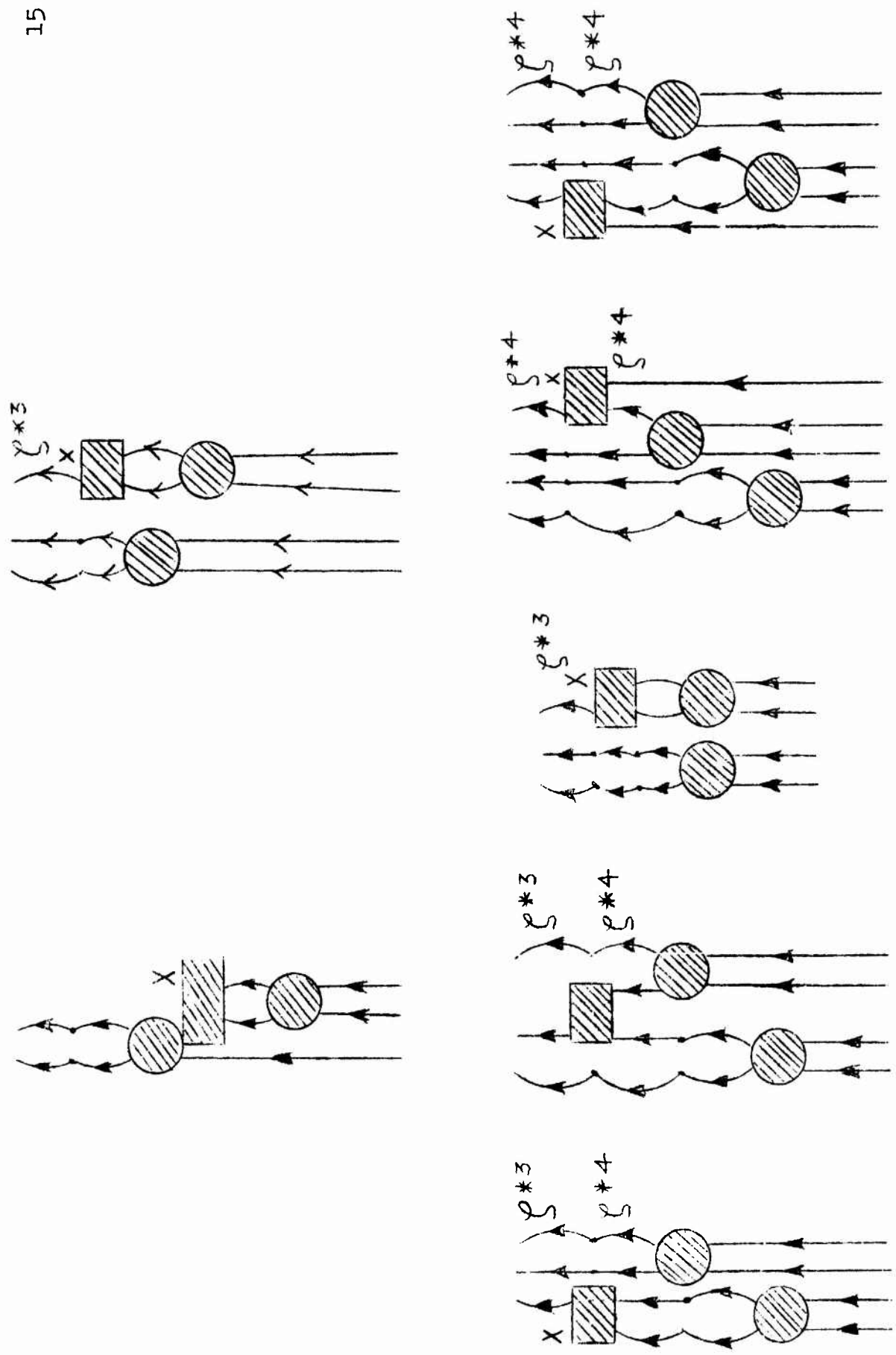


Figure 13(IV)

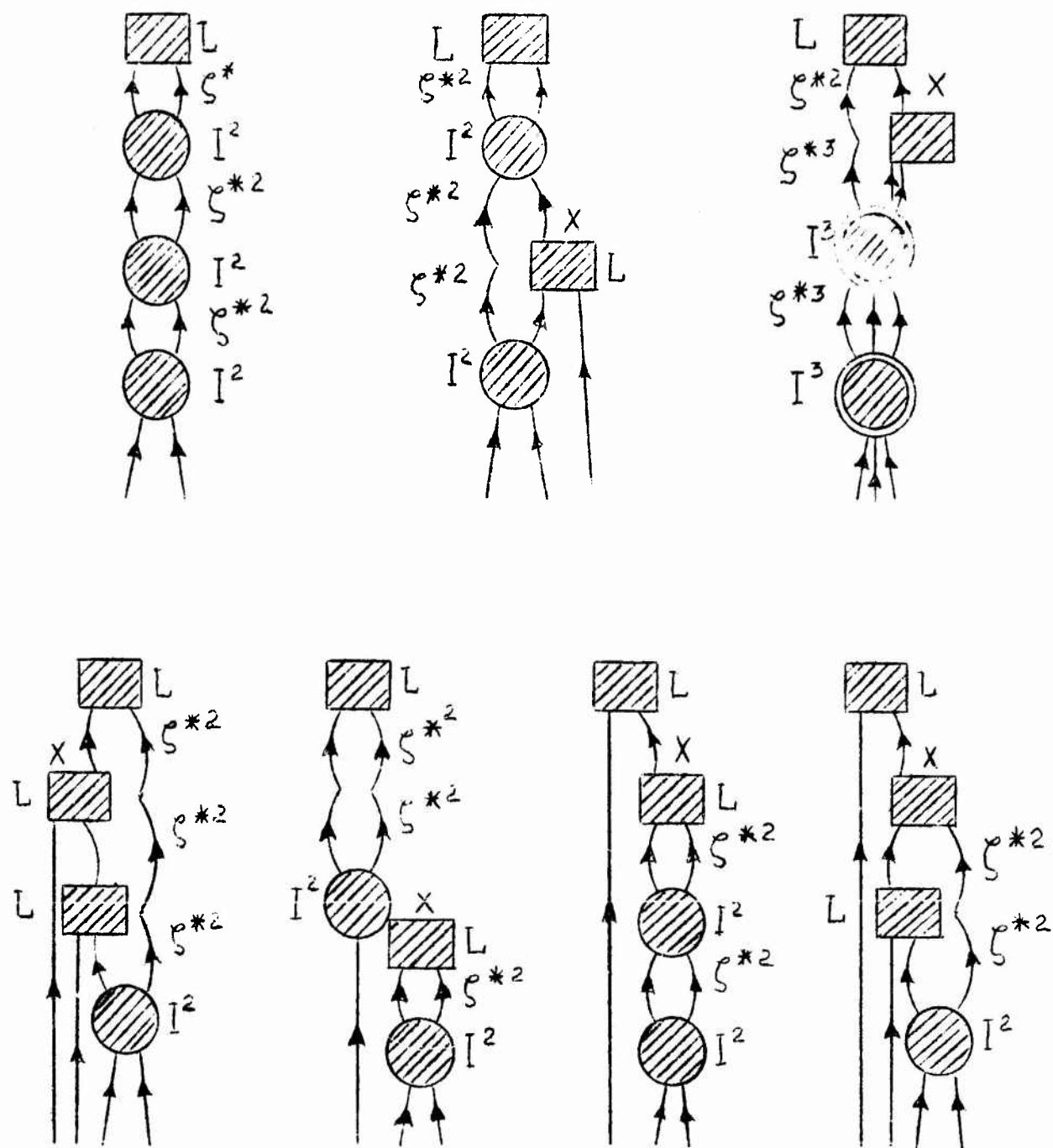


Figure 14

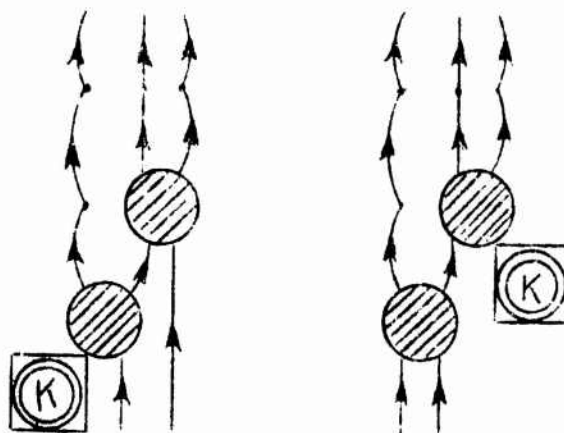


Figure 15

## IV. CONCLUSION

It has been suggested (4) that the divergence in the kinetic expansion can be remedied by a matching of the small relative velocity behavior where the divergences occur to terms introduced by expanding  $\underline{F}^2$  in fractional powers of  $\epsilon$ . For the simple initial value problem it is not possible to remedy the expansion in this way. This can be seen as follows.

Write

$$\varepsilon = \eta^v, \quad \chi_v = \eta^v t \quad (25)$$

$$\underline{F}^s = \sum_n \eta^n G^{sn} \quad (26)$$

The resulting perturbation equations reduce back to (3) with (5) and (6) for the simple initial value problem. If initial correlations are allowed, this reduction breaks down. It is an open question whether a choice of initial correlations can be made that makes the fractional power expansion equivalent to the pair kinetic equation.

The transient behavior of  $\underline{F}^s$  depends on the choice of initial conditions. We have therefore not discussed it in this paper since it is best understood in connection with the complete initial value problem.

## REFERENCES

- (1) G. Sandri, The New Foundations of Statistical Dynamics, Rutgers Univ. 1961-62. II edition.
- (2) Ibidem, Chapter VIII.
- (3) Ibidem, Chapter XII.
- (4) E. Frieman, M. Kruskal, and C. Su, private communication.

## APPENDIX

The Liouville equation is written as

$$\frac{\partial F^N}{\partial t} + H^N F^N = 0 \quad (A1)$$

with the energy operator

$$H^S = K^S + I^S \quad (A2)$$

The kinetic energy and interaction energy operators:

$$K^S = \sum_i^S \vec{v}_i \cdot \vec{v}_i \quad (A3)$$

$$I^S = \sum_{i < j}^S I_{ij}, \quad I_{ij} = \vec{v}_i U_{ij} \cdot \vec{v}_i + \vec{v}_j U_{ij} \cdot \vec{v}_j \quad (A4)$$

$$L_S = \sum_{i=1}^S L_{iS+1}, \quad L_{iS+1} = \int d\vec{x}_{S+1} d\vec{v}_{S+1} \vec{v}_i U_{iS+1} \cdot \vec{v}_i \quad (A5)$$

All quantities have been made dimensionless with respect to

- $r_0$  the range of the two body interaction
- $\phi_0$  the depth of the two body interaction
- $T$  the kinetic temperature of the gas

The weak-coupling parameter is

$$\epsilon = \frac{\phi_0}{kT} \ll 1 \quad (A6)$$

with the condition

$$N r_0^3 / V = 1 \quad (A7)$$

where  $N$  is the number of particles in the system and  $V$  is the volume of the box that encloses it.

The  $s$ -body distribution function is given by

$$F^S = \int F^N \frac{d\vec{x}_{S+1}}{V} \dots \frac{d\vec{x}_N}{V} d\vec{v}_{S+1} \dots d\vec{v}_N \quad (A8)$$